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Published in:
Proceedings of "Dynamics of Structures"

Publication date:
1994

Document Version
Accepted author manuscript, peer reviewed version

PERTURBATION SOLUTIONS FOR RANDOM LINEAR STRUCTURAL SYSTEMS SUBJECT TO RANDOM EXCITATION USING STOCHASTIC DIFFERENTIAL EQUATIONS

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SUMMARY

The paper deals with the first and second order statistical moments of the response of linear systems with random parameters subject to random excitation modelled as white-noise multiplied by an envelope function with random parameters. The method of analysis is basically a second order perturbation method using stochastic differential equations. The joint statistical moments entering the perturbation solution are determined by considering an augmented dynamic system with state variables made up of the displacement and velocity vector and their first and second derivatives with respect to the random parameters of the problem. Equations for partial derivatives are obtained from the partial differentiation of the equations of motion. The zero time-lag joint statistical moment equations for the augmented state vector are derived from the Itô differential formula. General formulation is given for multi-degree-of-freedom (MDOF) systems and the method is illustrated for a single-degree-of-freedom (SDOF) oscillator. The results are compared to those of exact results for a random oscillator subject to white noise excitation with random intensity.

1. INTRODUCTION

Structural uncertainties due to physical imperfections, model inaccuracies and system complexities are spatially distributed over the structure and can be mathematically modelled using either random variables or random processes which may be functions of time and/or space. The uncertainty of the structural model parameters and of the excitation parameters may induce uncertainty in the system response of the same magnitude as the random dynamic loads, and should therefore be included in the analysis. In the 1980s, the analysis of the response variability of stochastic structural systems recieved a lot of attention, consequently a new field, "Stochastic Finite Elements" was coined to stochastic mechanics. Although there have been papers on computationally expensive Monte Carlo solutions and reliability considerations, most of the studies done in stochastic finite elements have been on the second moment analysis of the response of stochastic systems under deterministic loading. The developments in this field are reviewed in [17,2,6,8,3]. This paper considers the approximate solution methods for the stochastic differential equations arising in stochastic finite elements for dynamic problems. In this respect, ordinary perturbations [9], mean-centered second-order perturbation solutions [15,11,5,12] and orthogonal polynomial expansions for the covariances [10] have been proposed as solution procedures. Divergent secular terms have been detected in the time domain analysis [9,15,12] and these are cured in the frequency domain [11,12]. The partial derivatives of the response processes with respect to random variables are proportional to time, hence perturbation solutions carry divergent secular terms and might blow up with time for undamped systems. For damped systems, since the divergent secular terms are under the governing control of the exponential damping decay, the existing deviations in the perturbation solutions become neither observable nor important with time [12].

For the random vibration of random systems, Chang and Yang [5] developed a mean-centered approximate second-order perturbation method in conjuction with modal expansion and an iterative scheme to solve non-linear dynamic problems for a beam element with structural uncertainities subject to Gaussian white noise excitation. They employed equivalent linearization with Gaussian closure to obtain equivalent linear system stiffness matrix and the local averaging method of Vanmarcke [18] to discretize random fields, thus, it is necessary to investigate the sensitivity of the response statistics to the density of the finite element mesh. Jensen and Iwan [10] employed an expansion of the covariances of the response in terms of a series of orthogonal polynomials that depend on the coefficients of the spectral decomposition of the uncertain parameters of the system. The accuracy of the approximation increases as the number of terms in the expansion is increased. Increase in the number of terms in the expansion is used to solve problems with high variabilities.

The authors have previously considered the stationary response of random linear elastic [13] as well as geometrically non-linear frame structures [14] subject to stationary random excitation based on a second order perturbation analysis. The random fields of the structure have been discretized by the weighted integral method of Deodatis [7] and Takada [16] in which Galerkin finite elements with deterministic shape functions are applied to stochastic differential equations. This provides consistency in the discretization of the random fields since the deterministic continuum and the random field are discretized by the same shape functions. The stochastic analysis of the geometrically non-linear structure was performed by an equivalent linearization approach in combination with a spectral approach. Since only the stationary response characteristics are to be computed, secular terms never arised. In both studies, the effects of the variability and the correlation length of the random fields of concern on the response are parametrically examined. Through comparisons with extensive Monte Carlo simulations and analytically available results, the second order perturbation method is observed to be a good approximation for variabilities up to 25-30 per cent.

Stochastic response of linear systems subject to white noise can be studied by means of Itô stochastic differential equations [1]. For random dynamic systems, the stochastic differential equations of motion have random coefficients. In this paper a second order perturbation method is developed for the stationary or non-stationary statistical moments of the response of MDOF structural systems subject to Gaussian white noise excitation multiplied by an envelope with random parameters, based on stochastic differential equations. The coefficients in the perturbation solution for the covariance are made up of time-dependent zero-time lag joint statistical moments of the responses and their first and second partial derivatives with respect to the random parameters. The necessary expectations are determined considering a sequence of augmented dynamic systems with the state vector made up of the displacement and velocity and their first and second derivatives with respect to the random parameters. This provides a very compact notation. Equations for the partial derivatives are obtained from the differentiation of the equations of motion.

The method is illustrated on a random linear SDOF oscillator subject to white noise excitation with random intensity. The obtained results have been compared to exact ones. The exact reults are obtained via the application of the total probability to the conditional analytical results.

2. LINEAR RANDOM MDOF STRUCTURAL SYSTEMS

The equations of motion of linear MDOF systems with random parameters subject to random excitation modelled as unit white noise multiplied by an envelope matrix with random parameters are

$$\mathbf{M}(\mathbf{X})\ddot{\mathbf{Y}}(\mathbf{X},t) + \mathbf{C}(\mathbf{X})\dot{\mathbf{Y}}(\mathbf{X},t) + \mathbf{K}(\mathbf{X})\mathbf{Y}(\mathbf{X},t) = \mathbf{Q}(\mathbf{X},t)\mathbf{W}(t) \tag{1}$$

where $\mathbf{M}(\mathbf{X})$, $\mathbf{C}(\mathbf{X})$ and $\mathbf{K}(\mathbf{X})$ are random mass, damping and stiffness matrices of dimension $p \times p$. $\mathbf{X}$ is a vector of dimension $d \times 1$ denoting all random variables of the structural and the load models. $\mathbf{Q}(\mathbf{X},t)$ is a matrix of dimension $p \times r$ indicating the envelope functions with random parameters. $\{\mathbf{W}(t), t \in]-\infty,\infty[\}$ is a vector of dimension $r \times 1$ of mutually independent unit white noise processes, i.e. a Gaussian process with the mean value and auto-covariance function as

$$E\big[W_\alpha(t)\big] = 0 \tag{2}$$

$$E\big[W_\alpha(t_1)W_\beta(t_2)\big] = \delta(t_1 - t_2)\delta_{\alpha\beta} \tag{3}$$

where $\delta(t_1 - t_2)$ signifies Dirac's delta function and $\delta_{\alpha\beta}$ is the Kronecker's delta. $\mathbf{X}^T = [X_1, \ldots, X_d]$ are zero-mean random variables with the covariances $\kappa_{X_iX_j}$, i.e.

$$E[X_i] = 0 \tag{4}$$

$$E[X_iX_j] = \kappa_{X_iX_j} \tag{5}$$

$X_1, \dots, X_d$, which will be referred to as the basic variables, are all assumed to be stochastically independent of the external excitation process $\{\mathbf{W}(t), t \in]-\infty, \infty[\}$.

Consider the following Taylor expansion of the structural random matrices and the excitation envelope matrix in terms of random parameters.

$$\left.\begin{aligned}
\mathbf{M}(\mathbf{X}) &= \mathbf{m}_0 + \mathbf{m}_i X_i + \frac{1}{2}\mathbf{m}_{ij} X_i X_j + \cdots \\
\mathbf{C}(\mathbf{X}) &= \mathbf{c}_0 + \mathbf{c}_i X_i + \frac{1}{2}\mathbf{c}_{ij} X_i X_j + \cdots \\
\mathbf{K}(\mathbf{X}) &= \mathbf{k}_0 + \mathbf{k}_i X_i + \frac{1}{2}\mathbf{k}_{ij} X_i X_j + \cdots \\
\mathbf{Q}(\mathbf{X}, t) &= \mathbf{q}_0(t) + \mathbf{q}_i(t) X_i + \frac{1}{2}\mathbf{q}_{ij}(t) X_i X_j + \cdots
\end{aligned}\right\} \tag{6}$$

where $\mathbf{m}_0 = \mathbf{M}(\mathbf{0})$, $\mathbf{m}_i = \frac{\partial}{\partial x_i}\mathbf{M}(\mathbf{0})$, $\mathbf{m}_{ij} = \frac{\partial^2}{\partial x_i x_j}\mathbf{M}(\mathbf{0})$, etc. and the summation convention is applied over the dummy indices $i, j = 1, 2, \dots, d$. $\mathbf{m}_0$, $\mathbf{c}_0$, $\mathbf{k}_0$, $\mathbf{m}_i$, $\mathbf{c}_i$, $\mathbf{k}_i$, $\mathbf{m}_{ij}$, $\mathbf{c}_{ij}$ and $\mathbf{k}_{ij}$ $\; i, j = 1, 2, \dots, d$ are deterministic constant matrices of dimension $p \times p$. $\mathbf{q}_0(t), \mathbf{q}_i(t)$ and $\mathbf{q}_{ij}(t)$ $\; i, j = 1, 2, \dots, d$ are deterministic matrices of dimension $p \times r$ which are functions of time.

The displacement response process $\{\mathbf{Y}(\mathbf{X}, t), t \in [0, \infty[\}$ and the velocity response process $\{\dot{\mathbf{Y}}(\mathbf{X}, t), t \in [0, \infty[\}$ of (1) are random partly because of the functional dependency of the external excitation process and partly because of the random basic variables $\mathbf{X}$. $\{\mathbf{Y}(\mathbf{0}, t), t \in [0, \infty[\}$ and $\{\dot{\mathbf{Y}}(\mathbf{0}, t), t \in [0, \infty[\}$ indicate the response process on condition of $\mathbf{X} = \mathbf{0}$, i.e. the stochastic displacement process obtained from (1) with mean values for the basic variables. Consider the following Taylor expansion of the stochastic system from the mean value system to the second order in the basic variables $X_1, \dots, X_d$

$$Y_m(\mathbf{X}, t) \simeq Y_m(\mathbf{0}, t) + Y_{m,x_i}(\mathbf{0}, t) X_i + \frac{1}{2} Y_{m,x_i x_j}(\mathbf{0}, t) X_i X_j + \cdots \quad m = 1, 2, \dots, p \tag{7}$$

$$\dot{Y}_m(\mathbf{X}, t) \simeq \dot{Y}_m(\mathbf{0}, t) + \dot{Y}_{m,x_i}(\mathbf{0}, t) X_i + \frac{1}{2} \dot{Y}_{m,x_i x_j}(\mathbf{0}, t) X_i X_j + \cdots \quad m = 1, 2, \dots, p \tag{8}$$

where $Y_{m,x_i}(\mathbf{0}, t) = \frac{\partial}{\partial x_i} Y_m(\mathbf{0}, t)$ etc. Further, the summation convention has been applied for the dummy indices $i, j = 1, \dots, d$. Use of (7), (8) and retaining terms up to the second order in the basic variables provides the following approximation for the unconditional covariance of the response processes.

$$\kappa_{Y_mY_n}(t) = E\big[Y_m(\mathbf{X},t)Y_n(\mathbf{X},t)\big] =$$

$$E\big[Y_m(0,t)Y_n(0,t)\big] + E\big[Y_m(0,t)Y_{n,x_i}(0,t)X_i\big] + E\big[Y_n(0,t)Y_{m,x_i}(0,t)X_i\big]+$$

$$E\Big[\big\{Y_{m,x_i}(0,t)Y_{n,x_j}(0,t) + \frac{1}{2}Y_m(0,t)Y_{n,x_ix_j}(0,t) + \frac{1}{2}Y_n(0,t)Y_{m,x_ix_j}(0,t)\big\}X_iX_j\Big] + \cdots$$

$$\simeq E\big[Y_m(0,t)Y_n(0,t)\big]+$$

$$\Big\{E\big[Y_{m,x_i}(0,t)Y_{n,x_j}(0,t)\big] + \frac{1}{2}E\big[Y_m(0,t)Y_{n,x_ix_j}(0,t)\big] + \frac{1}{2}E\big[Y_n(0,t)Y_{m,x_ix_j}(0,t)\big]\Big\}\kappa_{X_iX_j} \tag{9}$$

$$\kappa_{\dot{Y}_m\dot{Y}_n}(t) = E\big[\dot{Y}_m(\mathbf{X},t)\dot{Y}_n(\mathbf{X},t)\big] \simeq E\big[\dot{Y}_m(0,t)\dot{Y}_n(0,t)\big]+$$

$$\Big\{E\big[\dot{Y}_{m,x_i}(0,t)\dot{Y}_{n,x_j}(0,t)\big] + \frac{1}{2}E\big[\dot{Y}_m(0,t)\dot{Y}_{n,x_ix_j}(0,t)\big] + \frac{1}{2}E\big[\dot{Y}_n(0,t)\dot{Y}_{m,x_ix_j}(0,t)\big]\Big\}\kappa_{X_iX_j} \tag{10}$$

$$\kappa_{Y_m\dot{Y}_n}(t) = E\big[Y_m(\mathbf{X},t)\dot{Y}_n(\mathbf{X},t)\big] \simeq E\big[Y_m(0,t)\dot{Y}_n(0,t)\big]+$$

$$\Big\{E\big[Y_{m,x_i}(0,t)\dot{Y}_{n,x_j}(0,t)\big] + \frac{1}{2}E\big[Y_m(0,t)\dot{Y}_{n,x_ix_j}(0,t)\big] + \frac{1}{2}E\big[\dot{Y}_n(0,t)Y_{m,x_ix_j}(0,t)\big]\Big\}\kappa_{X_iX_j} \tag{11}$$

In deriving (9), (10) and (11) zero mean response has been assumed, i.e. $E\big[\mathbf{Y}(0,t)\big] = E\big[\dot{\mathbf{Y}}(0,t)\big] = \mathbf{0}$. Further, the independence of the basic variables on the white noise excitation processes is used.

In order to evaluate the expectations on the right hand sides of these solutions, stochastic differential equations must be formulated specifying the development of $\mathbf{Y}(0,t)$, $\dot{\mathbf{Y}}(0,t)$ and of the partial derivatives $\mathbf{Y}_{,x_i}(0,t)$, $\dot{\mathbf{Y}}_{,x_j}(0,t)$, $\mathbf{Y}_{,x_ix_j}(0,t)$, $\dot{\mathbf{Y}}_{,x_ix_j}(0,t)$. These are obtained from partial differentiation of (1) with respect the basic variables, evaluated at the mean structure $\mathbf{X} = \mathbf{0}$ and from the expansions listed in equation (4). A sufficient condition for the development of these state variables fulfilling the resulting equations is obtained if the mentioned state variables fulfil the following differential equations.

$$\mathbf{m}_0\ddot{\mathbf{Y}}(0,t) + \mathbf{c}_0\dot{\mathbf{Y}}(0,t) + \mathbf{k}_0\mathbf{Y}(0,t) = \mathbf{q}_0(t)\mathbf{W}(t) \tag{12}$$

$$\begin{aligned}
&\mathbf{m}_0\ddot{\mathbf{Y}}_{,x_i}(0,t) + \mathbf{c}_0\dot{\mathbf{Y}}_{,x_i}(0,t) + \mathbf{k}_0\mathbf{Y}_{,x_i}(0,t) = \\
&-\mathbf{m}_i\ddot{\mathbf{Y}}(0,t) - \mathbf{c}_i\dot{\mathbf{Y}}(0,t) - \mathbf{k}_i\mathbf{Y}(0,t) + \mathbf{q}_i(t)\mathbf{W}(t) = \\
&-\mathbf{m}_i\mathbf{m}_0^{-1}\big[-\mathbf{c}_0\dot{\mathbf{Y}}(0,t) - \mathbf{k}_0\mathbf{Y}(0,t) + \mathbf{q}_0(t)\mathbf{W}(t)\big] \\
&-\mathbf{c}_i\dot{\mathbf{Y}}(0,t) - \mathbf{k}_i\mathbf{Y}(0,t) + \mathbf{q}_i(t)\mathbf{W}(t) = \\
&\big(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{c}_0 - \mathbf{c}_i\big)\dot{\mathbf{Y}}(0,t) + \big(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{k}_0 - \mathbf{k}_i\big)\mathbf{Y}(0,t)+ \\
&\big(\mathbf{q}_i(t) - \mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{q}_0(t)\big)\mathbf{W}(t)
\end{aligned} \tag{13}$$

$$
\begin{aligned}
&\mathbf{m}_0\ddot{\mathbf{Y}}_{,x_ix_j}(0,t)+\mathbf{c}_0\dot{\mathbf{Y}}_{,x_ix_j}(0,t)+\mathbf{k}_0\mathbf{Y}_{,x_ix_j}(0,t)=\\
&-\mathbf{m}_i\ddot{\mathbf{Y}}_{,x_j}(0,t)-\mathbf{m}_j\ddot{\mathbf{Y}}_{,x_i}(0,t)-\mathbf{m}_{ij}\ddot{\mathbf{Y}}(0,t)-\mathbf{c}_i\dot{\mathbf{Y}}_{,x_j}(0,t)-\mathbf{c}_j\dot{\mathbf{Y}}_{,x_i}(0,t)-\\
&\mathbf{c}_{ij}\dot{\mathbf{Y}}(0,t)-\mathbf{k}_i\mathbf{Y}_{,x_j}(0,t)-\mathbf{k}_j\mathbf{Y}_{,x_i}(0,t)-\mathbf{k}_{ij}\mathbf{Y}(0,t)+\mathbf{q}_{ij}(t)\mathbf{W}(t)=\\
&-\mathbf{m}_i\mathbf{m}_0^{-1}\Big\{\big(\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{c}_0-\mathbf{c}_j\big)\dot{\mathbf{Y}}(0,t)+\big(\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{k}_0-\mathbf{k}_j\big)\mathbf{Y}(0,t)+\\
&\big(\mathbf{q}_j(t)-\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{q}_0(t)\big)\mathbf{W}(t)-\mathbf{c}_0\dot{\mathbf{Y}}_{,x_j}(0,t)-\mathbf{k}_0\mathbf{Y}_{,x_j}(0,t)\Big\}-\\
&\mathbf{m}_j\mathbf{m}_0^{-1}\Big\{\big(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{c}_0-\mathbf{c}_i\big)\dot{\mathbf{Y}}(0,t)+\big(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{k}_0-\mathbf{k}_i\big)\mathbf{Y}(0,t)+\\
&\big(\mathbf{q}_i(t)-\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{q}_0(t)\big)\mathbf{W}(t)-\mathbf{c}_0\dot{\mathbf{Y}}_{,x_i}(0,t)-\mathbf{k}_0\mathbf{Y}_{,x_i}(0,t)\Big\}-\\
&\mathbf{m}_{ij}\mathbf{m}_0^{-1}\big[-\mathbf{c}_0\dot{\mathbf{Y}}(0,t)-\mathbf{k}_0\mathbf{Y}(0,t)+\mathbf{q}_0(t)\mathbf{W}(t)\big]-\\
&\mathbf{c}_i\dot{\mathbf{Y}}_{,x_j}(0,t)-\mathbf{c}_j\dot{\mathbf{Y}}_{,x_i}(0,t)-\mathbf{c}_{ij}\dot{\mathbf{Y}}(0,t)-\\
&\mathbf{k}_i\mathbf{Y}_{,x_j}(0,t)-\mathbf{k}_j\mathbf{Y}_{,x_i}(0,t)-\mathbf{k}_{ij}\mathbf{Y}(0,t)+\mathbf{q}_{ij}(t)\mathbf{W}(t)
\end{aligned}
\tag{14}
$$

(12), (13) and (14) can next be combined into the following closed system of equivalent 1st order stochastic differential equations

$$
\dot{\mathbf{Z}}(t)=\mathbf{A}\mathbf{Z}(t)+\mathbf{b}(t)\mathbf{W}(t)\quad,\quad\mathbf{Z}(0)=\mathbf{0}
\tag{15}
$$

$$
\mathbf{Z}(t)=\begin{bmatrix}
\mathbf{Y}(0,t)\\
\dot{\mathbf{Y}}(0,t)\\
\mathbf{Y}_{,x_i}(0,t)\\
\dot{\mathbf{Y}}_{,x_i}(0,t)\\
\mathbf{Y}_{,x_j}(0,t)\\
\dot{\mathbf{Y}}_{,x_j}(0,t)\\
\mathbf{Y}_{,x_ix_j}(0,t)\\
\dot{\mathbf{Y}}_{,x_ix_j}(0,t)
\end{bmatrix}
$$

$$
\mathbf{b}(t)=\begin{bmatrix}
\mathbf{0}\\
\mathbf{m}_0^{-1}\mathbf{q}_0(t)\\
\mathbf{0}\\
-\mathbf{m}_0^{-1}(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{q}_0(t)-\mathbf{q}_i(t))\\
\mathbf{0}\\
-\mathbf{m}_0^{-1}(\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{q}_0(t)-\mathbf{q}_j(t))\\
\mathbf{0}\\
\mathbf{m}_0^{-1}\Big[(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{m}_j+\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{m}_i-\mathbf{m}_{ij})\mathbf{m}_0^{-1}\mathbf{q}_0(t)-\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{q}_i(t)-\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{q}_j(t)+\mathbf{q}_{ij}(t)\Big]
\end{bmatrix}
\tag{16}
$$

$$\mathbf{A}=\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{m}_0^{-1}\mathbf{k}_0 & -\mathbf{m}_0^{-1}\mathbf{c}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{H}_i & -\mathbf{m}_0^{-1}\mathbf{k}_0 & -\mathbf{m}_0^{-1}\mathbf{c}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_j & \mathbf{H}_j & \mathbf{0} & \mathbf{0} & -\mathbf{m}_0^{-1}\mathbf{k}_0 & -\mathbf{m}_0^{-1}\mathbf{c}_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{G}_{ij} & \mathbf{H}_{ij} & \mathbf{m}_0^{-1}\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{k}_0 & \mathbf{m}_0^{-1}\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{c}_0 & \mathbf{m}_0^{-1}\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{k}_0 & \mathbf{m}_0^{-1}\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{c}_0 & -\mathbf{m}_0^{-1}\mathbf{k}_0 & -\mathbf{m}_0^{-1}\mathbf{c}_0 \end{bmatrix} \tag{17}$$

$$\left.\begin{aligned} \mathbf{G}_i &= \mathbf{m}_0^{-1}(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{k}_0-\mathbf{k}_i) \\ \mathbf{H}_i &= \mathbf{m}_0^{-1}(\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{c}_0-\mathbf{c}_i) \\ \mathbf{G}_{ij} &= \mathbf{m}_0^{-1}[\mathbf{m}_j\mathbf{m}_0^{-1}(\mathbf{k}_i-\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{k}_0)+\mathbf{m}_i\mathbf{m}_0^{-1}(\mathbf{k}_j-\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{k}_0)+\mathbf{m}_{ij}\mathbf{m}_0^{-1}\mathbf{k}_0] \\ \mathbf{H}_{ij} &= \mathbf{m}_0^{-1}[\mathbf{m}_j\mathbf{m}_0^{-1}(\mathbf{c}_i-\mathbf{m}_i\mathbf{m}_0^{-1}\mathbf{c}_0)+\mathbf{m}_i\mathbf{m}_0^{-1}(\mathbf{c}_j-\mathbf{m}_j\mathbf{m}_0^{-1}\mathbf{c}_0)+\mathbf{m}_{ij}\mathbf{m}_0^{-1}\mathbf{c}_0] \end{aligned}\right\} \tag{18}$$

Zero initial conditions of the system have been assumed as shown in (15). The white noise excited system in (15) is the engineering interpretation of a Stratonovich differential system with linear drift-vector $\mathbf{AZ}(t)$ and diffusion vector $\mathbf{b}(t)$. Since the diffusion vector is state independent, the Stratonovich and Itô interpretation of equation (15) are identical with probability 1 [1].

The state vector $\mathbf{Z}(t)$ has the dimension $N = 2p + 4pd + 2pd^2$. If the basic variables are assumed to be uncorrelated, only the coefficients for $i = j$ in the inner sums on the right hand sides of (9), (10) and (11) contribute. Moreover, since partial derivatives with respect to i and j are identical, the coefficients can be determined from the differential system for the following reduced state state vector of the dimension $N = 6p$ for a fixed i.

$$\mathbf{Z}^T(t) = \left[\mathbf{Y}(0,t), \dot{\mathbf{Y}}(0,t), \mathbf{Y}_{,x_i}(0,t), \dot{\mathbf{Y}}_{,x_i}(0,t), \mathbf{Y}_{,x_ix_i}(0,t), \dot{\mathbf{Y}}_{,x_ix_i}(0.t)\right] \tag{19}$$

Here, the differential equations for the state vector (19) is obtained as a sub-system of (16).

3. DIFFERENTIAL EQUATIONS FOR STATISTICAL MOMENTS

(15) is a linear stochastic differential equation subject to Gaussian white noise, therefore, the state vector $\mathbf{Z}(t)$ is Gaussian. Further, the system has zero initial values, thus, $E[\mathbf{Z}(t)] = \mathbf{0}$. The covariances of $\mathbf{Z}(t)$ fully describe the joint probability density function of $\mathbf{Z}(t)$. Applying the Itô-formula and then performing the expectations, the following differential equations and associated initial values are obtained for the covariances, $\mu_{ij}(t) = E[Z_i(t)Z_j(t)]$, see e.g [1].

$$\dot{\mu}_{ij}(t) = A_{ik}\mu_{kj}(t) + A_{jk}\mu_{ki}(t) + b_i(t)b_j(t) \quad , \quad \mu_{ij}(0) = 0 \tag{20}$$

where summation convention is applied on dummy index $k = 1, 2, \ldots, N$.

4. LINEAR SDOF RANDOM OSCILLATOR

Next, the method is illustrated for a linear random SDOF oscillator subject to white noise excitation with random intensity. The equation of motion of a SDOF linear oscillator with random parameters subject to white noise excitation with random intensity is

$$M(\mathbf{X})\ddot{Y}(\mathbf{X},t) + C(\mathbf{X})\dot{Y}(\mathbf{X},t) + K(\mathbf{X})Y(\mathbf{X},t) = Q(\mathbf{X})W(t) \tag{21}$$

where $M(\mathbf{X})$, $C(\mathbf{X})$ and $K(\mathbf{X})$ are random mass, damping and stiffness. $Q(\mathbf{X})$ indicates the random intensity of the excitation which is taken as time-invariant in this problem. $\{W(t), t \in]-\infty, \infty[\}$ is a unit white noise process.

The four random parameters, entering the above equations are all modelled as random variables written on the following form.

$$\left.\begin{aligned} M(\mathbf{X}) &= m_0(1+X_1) \\ C(\mathbf{X}) &= c_0(1+X_2) \\ K(\mathbf{X}) &= k_0(1+X_3) \\ Q(\mathbf{X}) &= q_0(1+X_4) \end{aligned}\right\} \tag{22}$$

where $\mathbf{X}^T = [X_1, \ldots, X_4]$ are zero-mean random variables with the co-variances $\kappa_{X_iX_j}$. As previously, these random variables referred to as the basic variables are all assumed to be stochastically independent of the external excitation process $\{W(t), t \in]-\infty, \infty[\}$.

Equation (22) states that $m_1 = m_0$, $m_2 = m_3 = m_4 = 0$, $c_2 = c_0$, $c_1 = c_3 = c_4 = 0$, $k_3 = k_0$, $k_1 = k_2 = k_4 = 0$, $q_4 = q_0$, $q_1 = q_2 = q_3 = 0$ and $m_{ij} = c_{ij} = k_{ij} = q_{ij} = 0$ for $i, j = 1, 2, 3, 4$. Kronecker delta is introduced to eliminate the zero terms. For this oscillator problem, (15)-(18) then become

$$\dot{\mathbf{Z}}(t) = \mathbf{A}\mathbf{Z}(t) + \mathbf{b}W(t) \quad , \quad \mathbf{Z}(0) = \mathbf{0} \tag{23}$$

$$\mathbf{Z}(t) = \begin{bmatrix} Y(0,t) \\ \dot{Y}(0,t) \\ Y_{,x_i}(0,t) \\ \dot{Y}_{,x_i}(0,t) \\ Y_{,x_j}(0,t) \\ \dot{Y}_{,x_j}(0,t) \\ Y_{,x_ix_j}(0,t) \\ \dot{Y}_{,x_ix_j}(0,t) \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 0 \\ \frac{q_0}{m_0} \\ 0 \\ -\frac{q_0}{m_0}\delta_{1i} + \frac{q_0}{m_0}\delta_{4i} \\ 0 \\ -\frac{q_0}{m_0}\delta_{1j} + \frac{q_0}{m_0}\delta_{4j} \\ 0 \\ 2\frac{q_0}{m_0}\delta_{1i}\delta_{1j} - \frac{q_0}{m_0}\delta_{1j}\delta_{4i} - \frac{q_0}{m_0}\delta_{1i}\delta_{4j} \end{bmatrix} \tag{24}$$

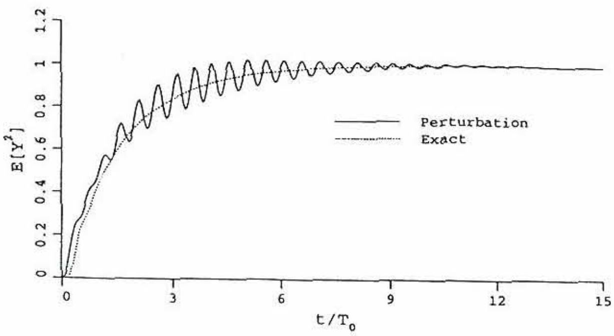

Figure 1. Unconditional variance of displacement, $\kappa_{YY}(t)$, versus undimensional time, $\frac{t}{T_0}$. Only M is random.

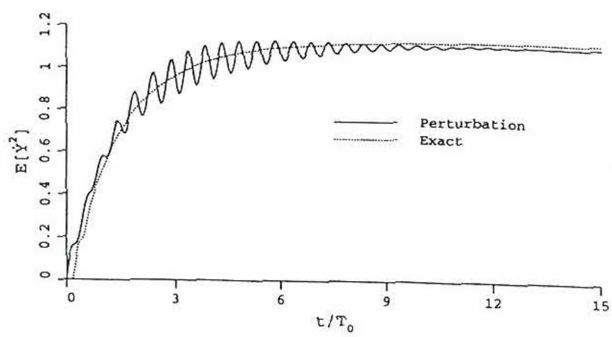

Figure 2. Unconditional variance of velocity, $\kappa_{\dot{Y}\dot{Y}}(t)$, versus undimensional time, $\frac{t}{T_0}$. Only M is random.

$$
\mathbf{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{k_0}{m_0} & -\frac{c_0}{k_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
G_i & H_i & -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
G_j & H_j & 0 & 0 & -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
G_{ij} & H_{ij} & \frac{k_0}{m_0}\delta_{1j} & \frac{c_0}{m_0}\delta_{1j} & \frac{k_0}{m_0}\delta_{1i} & \frac{c_0}{m_0}\delta_{1i} & -\frac{k_0}{m_0} & -\frac{c_0}{m_0}
\end{bmatrix} \tag{25}
$$

$$
\left.\begin{aligned}
G_i &= \frac{k_0}{m_0}(\delta_{1i} - \delta_{3i}) \\
H_i &= \frac{c_0}{m_0}(\delta_{1i} - \delta_{2i}) \\
G_{ij} &= \frac{k_0}{m_0}(-2\delta_{1i}\delta_{1j} + \delta_{1i}\delta_{3j} + \delta_{3i}\delta_{1j}) \\
H_{ij} &= \frac{c_0}{m_0}(-2\delta_{1i}\delta_{1j} + \delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j})
\end{aligned}\right\} \tag{26}
$$

5. NUMERICAL RESULTS

In what follows, a numerical example is worked out for an SDOF oscillator. The random parameters M, C, K and Q are assumed to be mutually stochastically independent and uniformly distributed, $M \sim U(a_M, b_M)$, $C \sim U(a_C, b_C)$, $K \sim U(a_K, b_K)$, $Q \sim U(a_Q, b_Q)$, where $a_M = E[M](1-\sqrt{3}V[M])$, $b_M = E[M](1+\sqrt{3}V[M])$, etc. The following mean values ($E[\cdot]$) and variational coefficients ($V[\cdot]$) are considered.

$$\left.\begin{array}{lll} E[M] = m_0 = 1.0 & , & V[M] = 0.3 \\ E[C] = c_0 = 0.1 & , & V[C] = 0.3 \\ E[K] = k_0 = 1.0 & , & V[\beta] = 0.3 \\ E[Q] = q_0 = \sqrt{2c_0k_0} & , & V[Q] = 0.3 \end{array}\right\} \tag{27}$$

The corresponding circular eigenfrequency, $\omega_0 = \sqrt{\frac{k_0}{m_0}}$, and damping ratio, $\zeta_0 = \frac{c_0}{2m_0\omega_0}$, of the mean oscillator are 1.0 and 0.05, respectively. The variance of the displacement and the velocity of the mean linear oscillator are both equal to 1. The variational coefficients of the parameters also indicate the standard deviation of the zero-mean basic variables X_i defined in (22). The proposed perturbation method is a very good approximation for small variabilities, i.e. small $V[\cdot]$. $V[\cdot] = 0.3$ is a rather high coefficient of variation, almost the limit of the proposed method. The results with $V[\cdot]$ are presented in figures 1-8 to show the performence of the method at such a large variability.

The linear statistical moment differential equations, listed in (20), are numerically solved by a 4th order Runge-Kutta scheme, with time step selected as $\Delta t = T_0/20$, where $T_0 = 2\pi$ is the eigenperiod of the mean linear oscillator. The results obtained by the present approximate second order perturbation method are compared to the exact ones in all the following figures. Since a joint probability function is assigned for the random variables, $f_{\mathbf{X}}(\mathbf{x})$, i.e.,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(b_M - a_M)(b_C - a_C)(b_K - a_K)(b_Q - a_Q)} \tag{28}$$

the exact unconditional non-stationary variances, $\kappa_{YY}(t)$ can be computed from the application of the total probability theorem on the conditional non-stationary variances, $\kappa_{YY}(\mathbf{X} = \mathbf{x})$, of the oscillator as

$$\kappa_{YY}(t) = \int_{\mathbf{x}} \kappa_{YY}(t \mid \mathbf{X} = \mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{29}$$

The non-stationary variances $\kappa_{YY}(t \mid \mathbf{X} = \mathbf{x})$ and $\kappa_{\dot{Y}\dot{Y}}(t \mid \mathbf{X} = \mathbf{x})$ on condition of the system $\mathbf{X} = \mathbf{x}$ are analytically available as follows

$$\kappa_{YY}(t \mid \mathbf{X} = \mathbf{x}) = \frac{q^2\left(-{\omega_d}^2 + e^{2\zeta\omega t}\,{\omega_d}^2 - \omega^2\,\zeta^2 + \omega^2\,\zeta^2\,\cos(2\omega_d t) - \omega\,\omega_d\,\zeta\,\sin(2\omega_d t)\right)}{4e^{2\zeta\omega t}\,m^2\omega^3\omega_d^2\zeta} \tag{30}$$

$$\kappa_{\dot{Y}\dot{Y}}(t \mid \mathbf{X} = \mathbf{x}) = \frac{q^2\left(-{\omega_d}^2 + e^{2\zeta\omega t}\,{\omega_d}^2 - \omega^2\,\zeta^2 + \omega^2\,\zeta^2\,\cos(2\omega_d t) + \omega\,\omega_d\,\zeta\,\sin(2\omega_d t)\right)}{4\,e^{2\zeta\omega t}\,m^2\,\omega\,{\omega_d}^2\,\zeta} \tag{31}$$

where $\omega = \sqrt{\frac{k}{m}}$, $\zeta = \frac{c}{2m\omega}$ and $\omega_d = \omega\sqrt{1-\zeta^2}$. The exact non-stationary unconditional variances presented in figures 1-8 are calculated from the numerical integration of (29) using equations (30) and (31).

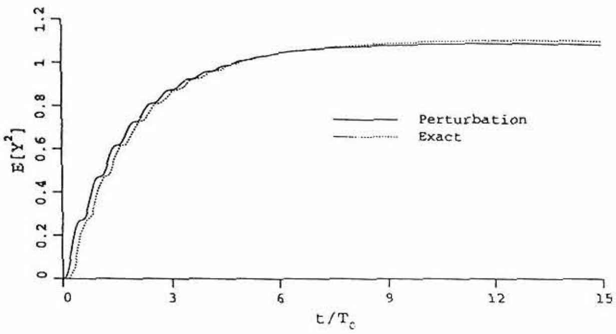

Figure 3. $\kappa_{YY}(t)$ versus $\frac{t}{T_0}$. Only C is random.

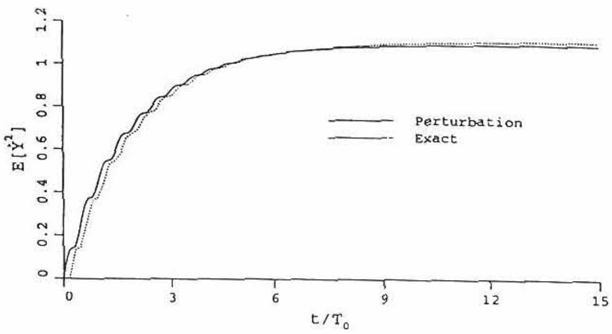

Figure 4. $\kappa_{\dot{Y}\dot{Y}}(t)$ versus $\frac{t}{T_0}$. Only C is random.

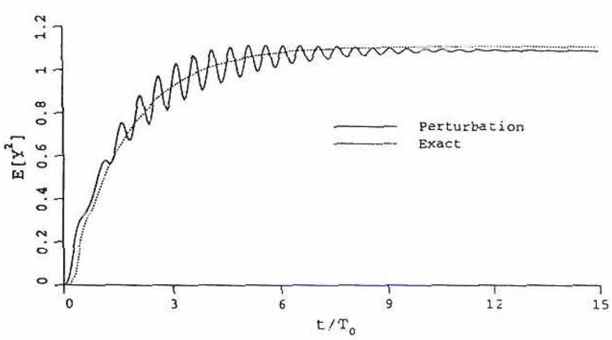

Figure 5. $\kappa_{YY}(t)$ versus $\frac{t}{T_0}$. Only K is random.

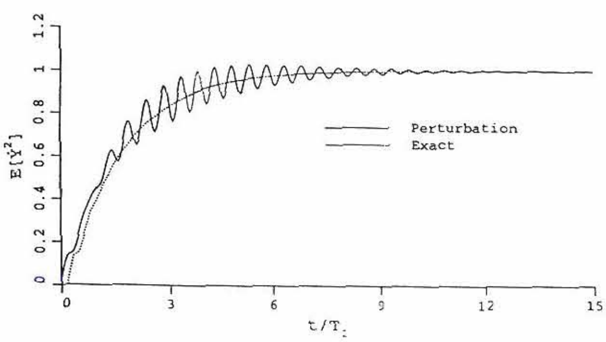

Figure 6. $\kappa_{\dot{Y}\dot{Y}}(t)$ versus $\frac{t}{T_0}$. Only K is random.

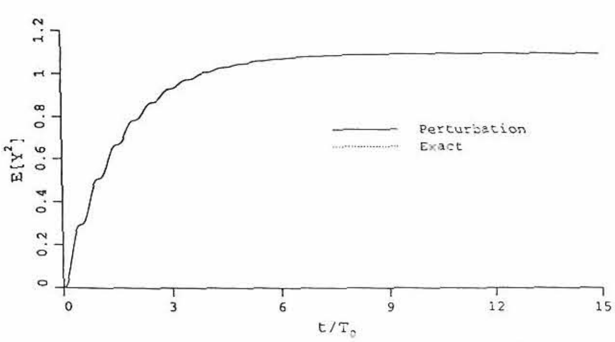

Figure 7. $\kappa_{YY}(t)$ versus $\frac{t}{T_0}$. Only Q is random.

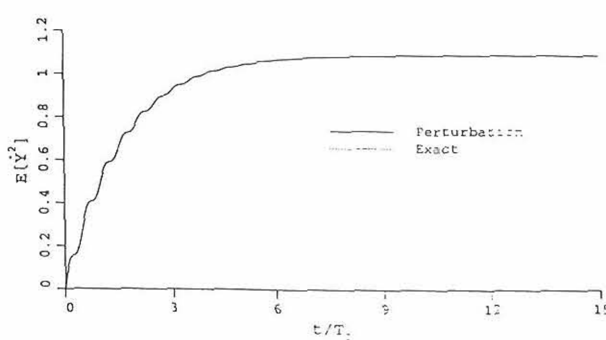

Figure 8. $\kappa_{\dot{Y}\dot{Y}}(t)$ versus $\frac{t}{T_0}$. Only Q is random.

As seen clearly from figures 1,2 and 5,6, the perturbation solution carry divergent secular terms in the non-stationary regime. Since the partial derivatives of the response processes with respect to random variables are proportional to time. To explain how these secular terms arise, a perturbation solution of order n for (30) and (31) from the mean structure can be performed. Then, terms of the types $t^n e^{-\zeta_0 \omega_0 t} \cos(w_d t)$ and $t^n e^{-\zeta_0 \omega_0 t} \sin(w_d t)$ will appear. Obviously, these terms will be dissipated as $t \to \infty$, but can be dominating at small t, especially if the damping is small. In the proposed second order perturbation method, the divergent terms are quadratic with time, t^2. Hence, extensions to higher order perturbations will not improve the solution for the non-stationary regime. The solutions would have blown up with time if there hadn't been any damping. For damped systems, since the divergent secular terms are under the governing control of the exponential damping decay, the existing deviations in the perturbation solutions become neither observable nor important at large t. Similar secular terms were previously detected by the authors in the perturbation solutions of linear oscillators with random stiffness subject to short duration earthquakes, [12]. On the other hand, for the studied example, no significant secular terms are present for only the random damping case, figures 3-4. The results are exact for only random white noise intensity case, figures 7-8. Further, the stationary results are estimated with high accuracy in all cases even for such high variabilities, $V[\cdot]$. The deviations between the perturbation results and the exact solutions in the stationary regime are due to latter's being dependent on the selected distribution (in this example, uniform distribution) whereas perturbation solutions are distribution free.

Analytical derivations yield the following exact stationary unconditional variances.

$$\kappa_{YY} = E\Big[\frac{Q^2}{2CK}\Big] = \frac{1}{6(b_C - a_C)(b_K - a_K)} \ln\Big(\frac{b_C}{a_C}\Big) \ln\Big(\frac{b_K}{a_K}\Big)(b_Q^2 + b_Q a_Q + a_Q^2) \quad (32)$$

$$\kappa_{\dot{Y}\dot{Y}} = E\Big[\frac{Q^2}{2MC}\Big] = \frac{1}{6(b_M - a_M)(b_C - a_C)} \ln\Big(\frac{b_M}{a_M}\Big) \ln\Big(\frac{b_C}{a_C}\Big)(b_Q^2 + b_Q a_Q + a_Q^2) \quad (33)$$

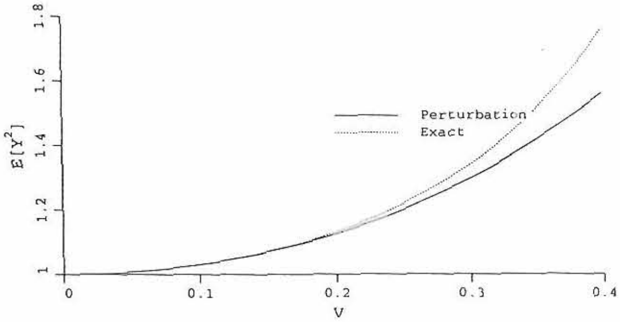

Figure 9. κ_{YY} versus V. All parameters are random with the same coefficient of variation V.

(32) is employed to illustrate the validity range of the second order perturbation solution for the stationary results in figure 9 where all random variables are given the same coefficient of variation, V. The stationary results for the perturbation solution can be obtained from the solution of linear equations listed in (20) with left hand sides equal to zero. From the comparison of exact stationary results and the second order perturbation results, it is concluded that the proposed second order perturbation method yields good results for variablities up to 25-30 per cent for stationary results.

6. CONCLUSIONS

A second order perturbation method using stochastic differential equations is developed for the stochastic response problem of linear systems with random parameters subject to random excitation modelled as white-noise multiplied by an envelope function with random parameters. The joint statistical moments entering the perturbation solution are determined by considering an augmented dynamic system with state variables made up of the displacement and velocity vector and their first and second derivatives with respect to the random parameters of the problem. Equations for partial derivatives are obtained from the partial differentiation of the equations of motion. The zero time-lag joint statistical moment equations for the augmented state vector are derived from the Itô differential formula. This provides a very compact formulation. Secular terms arise in the perturbation solution in the non-stationary phase until stationarity is attained in case of random mass and random stiffness paramaters. These are under the governing control of the structural damping of the system in the way that they are eventually dissipated.

From the studied numerical comparisons with the exact results for a random linear oscillator, it is concluded that the proposed second order perturbation method possesses secular divergent terms which are under the control of damping in the non-stationary regime and that it yields very good results for variablities up to 25-30 per cent for stationary response statistics.

7. ACKNOWLEDGEMENTS

The present research was partially supported by The Danish Technical Research Council within the research programme Dynamics of Structures.

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